

# The extremal genus embedding of graphs <sup>†</sup>

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## Abstract

Let  $W_n$  be a wheel graph with  $n$  spokes. How does the genus change if adding a degree-3 vertex  $v$ , which is not in  $V(W_n)$ , to the graph  $W_n$ ? In this paper, through the joint-tree model we obtain that the genus of  $W_n + v$  equals 0 if the three neighbors of  $v$  are in the same face boundary of  $\mathbb{P}(W_n)$ ; otherwise,  $\gamma(W_n + v) = 1$ , where  $\mathbb{P}(W_n)$  is the unique planar embedding of  $W_n$ . In addition, via the independent set, we provide a lower bound on the maximum genus of graphs, which may be better than both the result of D. Li & Y. Liu and the result of Z. Ouyang *etc.* in Europ. J. Combinatorics. Furthermore, we obtain a relation between the independence number and the maximum genus of graphs, and provide an algorithm to obtain the lower bound on the number of the distinct maximum genus embedding of the complete graph  $K_m$ , which, in some sense, improves the result of Y. Caro and S. Stahl respectively.

**Key Words:** joint-tree model; genus; maximum genus; independence number

**MSC(2000):** 05C10

## 1. Introduction

Graph considered here are all finite and connected. If the graph  $M$  can be obtained from a graph  $G$  by successively contracting edges and deleting edges and isolated vertices, then  $M$  is a *minor* of  $G$ . The minimum genus  $\gamma_{\min}(G)$  (or, simply, the genus  $\gamma(G)$ ) of a graph  $G$  is the minimum integer  $g$  such that there exists an embedding of  $G$  into the

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orientable surface  $S_g$  of genus  $g$ , and the *maximum genus*  $\gamma_M(G)$  of a connected graph  $G$  is the maximum integer  $k$  such that there exists an embedding of  $G$  into the orientable surface of genus  $k$ . The difference between the maximum genus and the minimum genus of a graph  $G$  is called the *genus range* of  $G$ . A graph  $G$  is said to be *upper embeddable* if  $\gamma_M(G) = \lfloor \frac{\beta(G)}{2} \rfloor$ , where  $\beta(G)$  is the *cycle rank* (or *Betti number*) of  $G$ . A *one-face* embedding (*two-face* embedding)  $\psi(G)$  of a graph  $G$  means that the face number of  $\psi(G)$  is one (two). An *odd vertex* is a vertex whose degree is an odd number. For  $n \geq 3$ , the *wheel* of  $n$  spokes is the graph  $W_n$  obtained from the  $n$ -cycle  $C_n$  by adding a new vertex (called the *center* of the *wheel*) and joint it to all vertices of  $C_n$ . For example,  $W_3 = K_4$ . A *subdivision* of an edge  $e \in E(W_n)$  means inserting a vertex of degree two to  $e$ , where the inserted vertex is called a *subdividing-vertex* of  $W_n$ . Let  $v$  be a degree-three vertex which is not in  $V(W_n)$ , then the graph  $W_n + v$ , which is called the *near-wheel* graph, means the connected graph obtained from  $W_n$  by joining  $v$  to  $v_i$  ( $i = 1, 2, 3$ ), where  $v_i$  may be a *subdividing-vertex* of  $W_n$  or a vertex which belongs to  $V(W_n)$ . Furthermore, the vertices  $v_1, v_2, v_3$  are called the *antennal-vertex* of the graph  $W_n + v$ .

Surfaces considered here are compact 2-dimensional manifold without boundary. An orientable surface  $S$  can be regarded as a polygon with even number of directed edges such that both  $a$  and  $a^-$  occurs once on  $S$  for each  $a \in S$ , where the power “ $-$ ” means that the direction of  $a^-$  is opposite to that of  $a$  on the polygon. For convenience, a polygon is represented by a linear sequence of lowercase letters. An elementary result in algebraic topology states that each orientable surface is equivalent to one of the following standard forms of surfaces:

$$O_p = \begin{cases} a_0 a_0^-, & p = 0, \\ \prod_{i=1}^p a_i b_i a_i^- b_i^-, & p \geq 1. \end{cases}$$

which are the sphere ( $p = 0$ ), torus ( $p = 1$ ), and the orientable surfaces of genus  $p$  ( $p \geq 2$ ). The genus of a surface  $S$  is denoted by  $g(S)$ . Let  $A, B, C, D$ , and  $E$  be possibly empty linear sequence of letters. Suppose  $A = a_1 a_2 \dots a_r$ ,  $r \geq 1$ , then  $A^- = a_r^- \dots a_2^- a_1^-$  is called the *inverse* of  $A$ . If  $\{a, b, a^-, b^-\}$  appear in a sequence of the form of  $AaBbCa^-Db^-E$ , then they are said to be an *interlaced set*; otherwise, a *parallel set*. Let  $\tilde{S}$  be the set of all surfaces. For a surface  $S \in \tilde{S}$ , we obtain its genus  $g(S)$  by using the following transforms to determine its equivalence to one of the standard forms.

**Transform 1**  $Aaa^- \sim A$ , where  $A \in \tilde{S}$  and  $a \notin A$ .

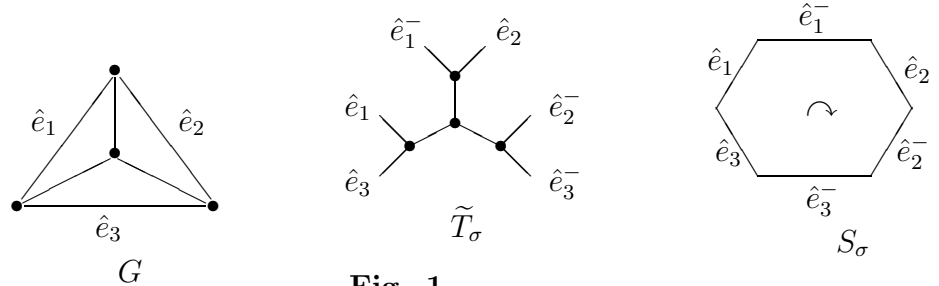
**Transform 2**  $AabBb^-a^- \sim AcBc^-$ .

**Transform 3**  $(Aa)(a^-B) \sim (AB)$ .

**Transform 4**  $AaBbCa^-Db^-E \sim ADCBEaba^-b^-$ .

In the above transforms, the parentheses stand for cyclic order. For convenience, the parentheses are always omitted when unnecessary to distinguish cyclic or linear order. For more details concerning surfaces, the reader is referred to [1] and [2].

Let  $T$  be a spanning tree of a graph  $G = (V, E)$ , then  $E = E_T + \hat{E}_T$ , where  $E_T$  consists of all the tree edges, and  $\hat{E}_T = \{\hat{e}_1, \hat{e}_2, \dots, \hat{e}_\beta\}$  consists of all the co-tree edges, where  $\beta = \beta(G)$  is the cycle rank of  $G$ . Split each co-tree edge  $\hat{e}_i = (u[\hat{e}_i], v[\hat{e}_i]) \in \hat{E}_T$  into two semi-edges  $(u[\hat{e}_i], v_i)$ ,  $(v[\hat{e}_i], \bar{v}_i)$ , denoted by  $\hat{e}_i^+$  and  $\hat{e}_i^-$  respectively. Let  $\tilde{T} = (V + V_1, E + E_1)$ , where  $V_1 = \{v_i, \bar{v}_i | 1 \leq i \leq \beta\}$ ,  $E_1 = \{(u[\hat{e}_i], v_i), (v[\hat{e}_i], \bar{v}_i) | 1 \leq i \leq \beta\}$ . Obviously,  $\tilde{T}$  is a tree. A rotation at a vertex  $v$ , which is denoted by  $\sigma_v$ , is a cyclic permutation of edges incident on  $v$ . A rotation system  $\sigma = \sigma_G$  for a graph  $G$  is a set  $\{\sigma_v | \forall v \in V(G)\}$ . The tree  $\tilde{T}$  with a rotation system of  $G$  is called a *joint-tree* of  $G$ , and is denoted by  $\tilde{T}_\sigma$ . Because  $\tilde{T}_\sigma$  is a tree, it can be embedded in the plane. By reading the lettered semi-edges of  $\tilde{T}_\sigma$  in a fixed direction (clockwise or anticlockwise), we can get an algebraic representation of the surface which is represented by a  $2\beta$ -polygon. Such a surface, which is denoted by  $S_\sigma$ , is called an associated surface of  $\tilde{T}_\sigma$ . A joint-tree  $\tilde{T}_\sigma$  of  $G$  and its associated surface is illustrated by Fig.1, where the rotation at each vertex of  $G$  complies with the clockwise rotation. From [1], there is 1-1 correspondence between the associated surfaces (or joint-trees) and the embeddings of a graph. The *joint-tree* is originated from the early works of Liu [3], and more detailed information about the *joint-tree* can be found in [1]. Terminologies and notations not defined here can be seen in [4] for graph theory and [5] for topological graph theory.



**Fig. 1.**

The following lemma is essential in the whole paper.

**Lemma 1.1** <sup>[6]</sup> Every simple 3-connected planar graph has a unique planar embedding.

**Lemma 1.2** The minimum genus of a minor of a graph  $G$  can never be larger than  $\gamma(G)$ .

**Proof** Let the graph  $G$  be embedded in a surface  $S$ , then contracting an edge  $e$  of  $G$  on  $S$  can obtain an embedding of the contracted graph  $G/e$  on  $S$ . Moreover, edge deletion can never increase embedding genus. Thus, the lemma is obtained.  $\square$

**Lemma 1.3** If an orientable surface  $S$  has the form as  $(AxByCx^-Dy^-E)$ , then  $g(S) \geq 1$ , furthermore, the genus of  $S$  is  $p(\geq 1)$  if, and only if,  $ADCBE$  is with genus  $p - 1$ .

**Proof** According to the Transform 4, it is obvious.  $\square$

## 2. The genus of the near-wheel graphs

It is obvious that  $W_n$  is 3-connected and  $\gamma(W_n) = 0$ . So, according to Lemma 1.1,  $W_n$  has an unique embedding in the plane. We denote this unique planar embedding of  $W_n$  by  $\mathbb{P}(W_n)$ .

**Lemma 2.1** Let  $\mathbb{P}(W_n)$  be the planar embedding of the wheel  $W_n$  with  $n$  spokes,  $v$  be a degree-three vertex which is not in  $W_n$ , then the genus  $\gamma(W_n + v)$  of the graph  $W_n + v$  equals 0 if the three *antennal vertices* of  $W_n + v$  are in the same face boundary of  $\mathbb{P}(W_n)$ .

**Proof** Let  $v_1, v_2, v_3$  be the three *antennal vertices* of  $W_n + v$ ,  $f_1$  be the face of  $\mathbb{P}(W_n)$  with  $v_1, v_2, v_3$  on it, then we can get a planar embedding of  $W_n + v$  by placing  $v$  in the interior of  $f_1$  and jointing  $vv_i$  ( $i = 1, 2, 3$ ).  $\square$

**Lemma 2.2** Let  $\mathbb{P}(W_n)$  be the planar embedding of the wheel  $W_n$  with  $n$  spokes,  $v$  be a degree-three vertex which is not in  $W_n$ , then the genus  $\gamma(W_n + v)$  of the graph  $W_n + v$  equals 1 if the following two conditions are satisfied: (i) the three *antennal-vertex* of  $W_n + v$  are in the boundary of two different faces of  $\mathbb{P}(W_n)$ ; (ii) there is no face of  $\mathbb{P}(W_n)$  whose boundary contains all the three *antennal-vertex*.

**Proof** It is easy to find out that  $K_{3,3}$  is a minor of  $W_n + v$ . According to Lemma 1.2 we can get that  $\gamma(W_n + v) \geq 1$ .

Let  $v_1, v_2, v_3$  be the three *antennal-vertex* of  $W_n + v$ . Because the three *antennal-vertex* of  $W_n + v$  are in the boundary of two different faces of  $\mathbb{P}(W_n)$ , without loss of generality, we may assume that  $v_1, v_2$  are in the boundary of  $f_1$ , and  $v_3$  in  $f_2$ , where  $f_1$  and  $f_2$  are two different faces of  $\mathbb{P}(W_n)$ . Putting  $v$  in the interior of  $f_1$  and joining  $vv_i$  ( $i = 1, 2, 3$ ), then we will get a torus embedding of  $W_n + v$  if add a handle to the plane with the edge  $vv_3$  on it. So  $\gamma(W_n + v) \leq 1$ .

From the above we can get that  $\gamma(W_n + v) \geq 1$  and  $\gamma(W_n + v) \leq 1$ . So  $\gamma(W_n + v) = 1$ .  $\square$

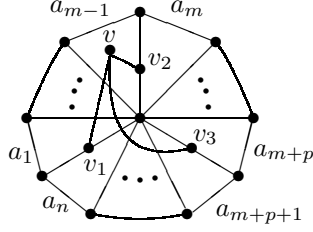
**Lemma 2.3** Let  $\mathbb{P}(W_n)$  be the planar embedding of the wheel  $W_n$  with  $n$  spokes,  $v$  be a degree-three vertex which is not in  $W_n$ , then the genus  $\gamma(W_n + v)$  of the graph  $W_n + v$  equals 1 if any pair of the three *antennal-vertex* of  $W_n + v$  are not in a same face boundary of  $\mathbb{P}(W_n)$ .

**Proof** It is not difficult to find out that  $K_{3,3}$  is a minor of  $W_n + v$ . According to Lemma 1.2 we can get that  $\gamma(W_n + v) \geq 1$ .

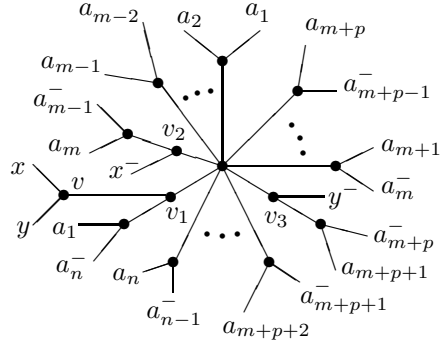
**Case 1:** The three *antennal-vertex* of  $W_n + v$  are all *subdividing-vertex* of  $W_n$ .

Let  $v_1, v_2, v_3$  be the three *antennal-vertex* of  $W_n + v$ . For any pair of the three *antennal-vertex* of  $W_n + v$  are not in a same face boundary of  $\mathbb{P}(W_n)$ , the vertices  $v_1, v_2$  and  $v_3$  must belong to one of the following two subcases: (1)  $v_1, v_2$  and  $v_3$  are in three

different spokes of  $W_n$ , furthermore, any pair of these three spokes are not in a same face boundary of  $\mathbb{P}(W_n)$ ; (2) one of  $\{v_1, v_2, v_3\}$  is on the boundary of the unbounded face of  $\mathbb{P}(W_n)$ , and the other two are in two different spokes of  $\mathbb{P}(W_n)$ , where the two spokes are not on a same face boundary of  $\mathbb{P}(W_n)$ .



**Fig.2:**  $W_n + v$



**Fig.3:**  $\tilde{T}_\sigma$

In the first subcase, the graph  $W_n + v$  and one of its joint-tree are shown in Fig.2 and Fig.3 respectively, where we denoted the edge  $(v, v_2)$  by  $x$ , and  $(v, v_3)$  by  $y$ . In Fig.2, the edges of the  $n$ -cycle in  $W_n$ , according to the clockwise rotation, are denoted by  $a_1, a_2, \dots, a_n$ . The surface associated with the joint-tree in Fig.3 is

$$\begin{aligned} S &= a_1 y x x^- a_m a_{m-1}^- a_{m-1}^- a_{m-2}^- a_{m-2}^- \dots a_2^- a_2^- a_1^- a_{m+p} a_{m+p-1}^- a_{m+p-1}^- a_{m+p-2}^- \\ &\quad a_{m+p-2}^- \dots a_{m+1}^- a_{m+1}^- a_m^- y^- a_{m+p} a_{m+p+1}^- a_{m+p+1}^- a_{m+p+2}^- a_{m+p+2}^- \dots a_n a_n^- \\ &\sim a_1 y a_m a_1^- a_{m+p} a_m^- y^- a_{m+p}^- \\ &\sim a_{m+p} a_m^- a_m a_{m+p}^- a_1 y a_1^- y^- \\ &\sim a_1 y a_1^- y^- \end{aligned}$$

Obviously,  $g(S) = 1$ . So  $\gamma(W_n + v) \leq 1$ . On the other hand  $\gamma(W_n + v) \geq 1$ . Therefore, in the first subcase,  $\gamma(W_n + v) = 1$ .

In the second subcase, the graph  $W_n + v$  and one of its joint-tree are shown in Fig.4 and Fig.5 respectively, where we denoted the edge  $(v, v_2)$  by  $x$ , and  $(v, v_3)$  by  $y$ . In Fig.4, the edges of the  $n$ -cycle in  $W_n$ , according to the clockwise rotation, are denoted by  $a_1, a_2, \dots, a_{m-1}, b, a_m, \dots, a_n$ . The surface associated with the joint-tree in Fig.5 is

$$\begin{aligned} S &= a_1 y x x^- a_m a_{m-1}^- a_{m-1}^- a_{m-2}^- a_{m-2}^- \dots a_2^- a_2^- a_1^- a_{m+p} a_{m+p-1}^- a_{m+p-1}^- a_{m+p-2}^- \\ &\quad a_{m+p-2}^- \dots a_{m+1}^- a_{m+1}^- a_m^- y^- a_{m+p} a_{m+p+1}^- a_{m+p+1}^- a_{m+p+2}^- a_{m+p+2}^- \dots a_n a_n^- \\ &\sim a_1 y a_m a_1^- a_{m+p} a_m^- y^- a_{m+p}^- \\ &\sim a_{m+p} a_m^- a_m a_{m+p}^- a_1 y a_1^- y^- \\ &\sim a_1 y a_1^- y^- \end{aligned}$$

Obviously,  $g(S) = 1$ . So  $\gamma(W_n + v) \leq 1$ . On the other hand  $\gamma(W_n + v) \geq 1$ . Therefore, in the second subcase,  $\gamma(W_n + v) = 1$ .



**Lemma 3.1** <sup>[7]</sup> The maximum genus of a cubic graph  $G$  equals the cardinality of the maximum non-separating independent set of  $G$ .

But for general graphs that is not necessary cubic, there is no result concerning the maximum genus which is characterized by the independent set of the graph. In the following, we will provide a lower bound of the maximum genus, which is characterized via the independent set, for general graphs. Furthermore, there are examples shown that the bound may be tight, and, in some sense, may be better than the result obtained by Li and Liu<sup>[8]</sup>, and the result obtained by Z. Ouyang *etc.*<sup>[9]</sup>.

**Theorem B** Let  $G$  be a connected graph whose minimum degree is at least 3. If  $A = \{v_1, v_2, \dots, v_m\}$  is an independent set such that  $G - A$  is connected, then

$$\gamma_M(G) \geq \frac{1}{2} \sum_{i=1}^m (d(v_i) - \varepsilon_i) + \gamma_M(G - \{v_1, v_2, \dots, v_m\}),$$

where for each index  $i$  ( $1 \leq i \leq m$ ),  $\varepsilon_i = 1$  if  $d(v_i) \equiv 1 \pmod{2}$  and  $\varepsilon_i = 2$  otherwise.

**Proof** Without loss of generality, let  $H$  be the graph obtained from  $G$  by successively deleting  $v_1, v_2, \dots, v_m$  from  $G$ , and  $\psi(H)$  be a maximum genus embedding of  $H$ . We first add the vertex  $v_m$  to  $H$ .

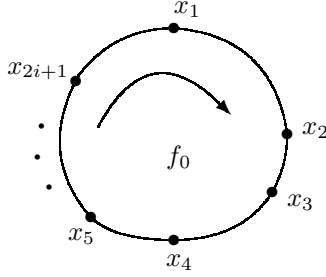
**Case 1:**  $d_G(v_m) \equiv 1 \pmod{2}$ .

Without loss of generality, let  $d_G(v_m) = 2i + 1$ , and  $x_1, x_2, \dots, x_{2i+1}$  be the  $2i + 1$  neighbors of  $v_m$  in  $G$ . According to the  $2i + 1$  neighbors of  $v_m$  are in the same face boundary of  $\psi(H)$  or not, we will discuss in the following two subcases.

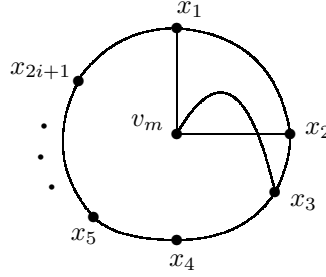
**Subcase 1.1:** All the neighbors of  $v_m$  are in the same face boundary of  $\psi(H)$ .

Let  $f_0$ , which is bounded by  $B_0$ , be the face of  $\psi(H)$  that  $x_1, x_2, \dots, x_{2i+1}$  are on the boundary of it. Firstly, we put  $v_m$  in  $f_0$  and connect each of  $\{x_1, x_2, x_3\}$  to  $v_m$ , and denote this resulting graph by  $H_1$ . Through the manner depicted by Fig.7, where each *vertex-rotation* is the same with that of  $\psi(H)$  except  $v_m$ , we can get an embedding  $\psi(H_1)$  of  $H_1$  such that its face number is the same with that of  $\psi(H)$ . From the equation  $V - E + F = 2 - 2g$ , it can be easily deduced that the maximum genus of  $H_1$  is at least one more than that of  $H$ .

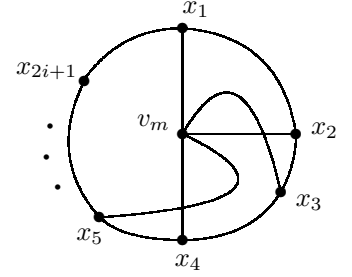
Now connect each of  $\{x_4, x_5\}$  to  $v_m$ , and denote the resulting graph by  $H_2$ . Through the manner depicted by Fig.8, we can get an embedding  $\psi(H_2)$  of  $H_2$ , which has the same face number with that of  $\psi(H)$ . From the equation  $V - E + F = 2 - 2g$ , it can be easily deduced that the maximum genus of  $H_2$  is at least two more than that of  $H$ .



**Fig.6:**  $B_0$



**Fig.7:**  $\psi(H_1)$

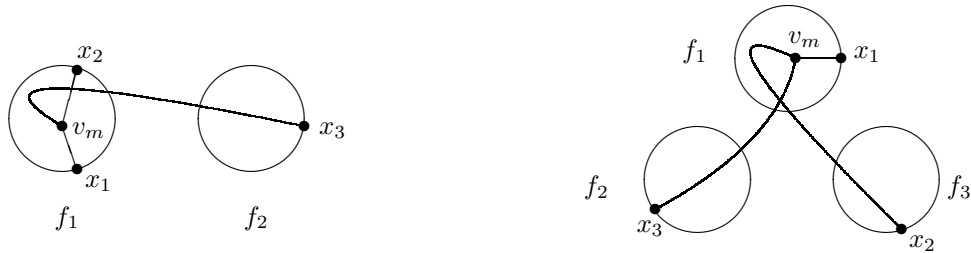


**Fig.8:**  $\psi(H_2)$

Similar to the manner of connecting  $\{x_4, x_5\}$  to  $v_m$ , we can connect  $\{x_6, x_7\}, \dots, \{x_{2i}, x_{2i+1}\}$  to  $v_m$ . Eventually, we will get an embedding of  $H + v_m$ . It can be easily deduced that the maximum genus of  $H + v_m$  is at least  $\frac{1}{2}(d(v_m) - 1) + \gamma_M(G - \{v_1, v_2, \dots, v_m\})$ .

**Subcase 1.2:** There is no face boundary of  $\psi(H)$  containing all the neighbors of  $v_m$ .

First, add  $v_m$  to  $H$  and connect each of  $\{x_1, x_2, x_3\}$  to  $v_m$ . The resulting graph is denoted by  $H_1$ . If  $x_1, x_2, x_3$  are in two different face boundaries of  $\psi(H)$ , say  $f_1$  and  $f_2$ , then via the manner depicted by the left part of Fig.9, we can get an embedding  $\psi(H_1)$  of  $H_1$  whose face number is the same with that of  $\psi(H)$ . If  $x_1, x_2, x_3$  are in three different face boundaries of  $\psi(H)$ , say  $f_1, f_2$ , and  $f_3$ , then through the manner depicted by the right part of Fig.9, we can get an embedding  $\psi(H_1)$  of  $H_1$  whose face number is two less than that of  $\psi(H)$ . From the equation  $V - E + F = 2 - 2g$ , it can be easily deduced that the maximum genus of  $H_1$  is at least one more than that of  $H$ .



**Fig.9**

Similarly, connect  $\{x_4, x_5\}, \dots, \{x_{2i}, x_{2i+1}\}$  to  $v_m$ . Eventually, we will get an embedding of  $H + v_m$ , and it can be easily deduced that the maximum genus of  $H + v_m$  is at least  $\frac{1}{2}(d(v_m) - 1) + \gamma_M(G - \{v_1, v_2, \dots, v_m\})$ .

From Subcase 1.1 and Subcase 1.2 we can get that if  $d_G(v_m) \equiv 1 \pmod{2}$ , then  $\gamma_M(H + v_m) \geq \frac{1}{2}(d(v_m) - 1) + \gamma_M(G - \{v_1, v_2, \dots, v_m\})$ .

**Case 2:**  $d_G(v_m) \equiv 0 \pmod{2}$ .

Similar to that of Case 1, we can get that if  $d_G(v_m) \equiv 0 \pmod{2}$ , then  $\gamma_M(H + v_m) \geq \frac{1}{2}(d(v_m) - 2) + \gamma_M(G - \{v_1, v_2, \dots, v_m\})$ .



From Case 1 and Case 2 we can get that  $\gamma_M(H + v_m) \geq \frac{1}{2}(d(v_m) - \varepsilon_i) + \gamma_M(H)$ , where  $\varepsilon_i = 1$  if  $d(v_i) \equiv 1 \pmod{2}$  and  $\varepsilon_i = 2$  otherwise.

Similarly to that of  $v_m$ , we can add  $v_{m-1}, v_{m-2}, \dots, v_1$ , one by one, to  $H + v_m$ . Eventually we will get an embedding of  $G$ , and it is not hard to obtain that the maximum genus of  $G$  is at least  $\frac{1}{2} \sum_{i=1}^m (d(v_i) - \varepsilon_i) + \gamma_M(G - \{v_1, v_2, \dots, v_m\})$ , where for each index  $i (1 \leq i \leq m)$ ,  $\varepsilon_i = 1$  if  $d(v_i) \equiv 1 \pmod{2}$  and  $\varepsilon_i = 2$  otherwise.  $\square$

Noticing that the upper embeddability of a graph would not be changed if adding an odd vertex to it, we can get the following theorem whose proof is similar to that of Theorem B.

**Theorem C** Let  $G$  be a connected graph and  $A_1, A_2, \dots, A_s$  be a sequence of disjoint independent vertex sets which satisfy: (i)  $G_0 = G$ ,  $G_i = G_{i-1} - A_i$  is connected ( $i = 1, 2, \dots, s$ ); (ii) each vertex of  $A_i$  ( $i = 1, 2, \dots, s$ ) is an odd vertex in  $G_{i-1}$ . Then for  $i = 0, 1, \dots, s-1$ ,

$$\gamma_M(G_i) \geq \frac{1}{2} \sum_{v \in A_{i+1}} (d_{G_i}(v) - 1) + \gamma_M(G_{i+1}).$$

In particular, if one of the graph sequence  $G_1, G_2, \dots, G_s$  is upper embeddable, then  $G$  is upper embeddable.

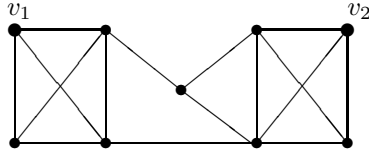
**Remark** In 2000, through the girth  $g$  and connectivity of graphs, D. Li and Y. Liu<sup>[8]</sup> obtained the lower bound of the maximum genus of graphs, which is displayed by the following table, where the first row and the first column represents the girth and connectivity respectively.

	$g=3$	$g=4$	$g=5$	$g=6$	$g=7$	$g=8$	$g=9$	$g=10$	$g=12$
1	$\frac{\beta(G)+2}{3}$	$\frac{\beta(G)+2}{4}$	$\frac{2\beta(G)+2}{5}$	$\frac{3\beta(G)+2}{7}$	$\frac{5\beta(G)+2}{11}$	$\frac{7\beta(G)+2}{15}$	$\frac{14\beta(G)+2}{29}$	$\frac{17\beta(G)+2}{35}$	$\frac{31\beta(G)+2}{63}$
2	$\frac{\beta(G)+2}{3}$	$\frac{\beta(G)+2}{4}$	$\frac{2\beta(G)+3}{5}$	$\frac{3\beta(G)+4}{7}$	$\frac{6\beta(G)+7}{11}$	$\frac{7\beta(G)+8}{15}$	$\frac{14\beta(G)+15}{29}$	$\frac{17\beta(G)+18}{35}$	$\frac{31\beta(G)+32}{63}$
3	$\frac{\beta(G)+2}{3}$	$\frac{3\beta(G)+4}{7}$	$\frac{5\beta(G)+6}{11}$	$\frac{7\beta(G)+8}{15}$	$\frac{11\beta(G)+12}{23}$	$\frac{15\beta(G)+16}{31}$	$\frac{29\beta(G)+30}{59}$	$\frac{35\beta(G)+36}{71}$	$\frac{63\beta(G)+64}{127}$

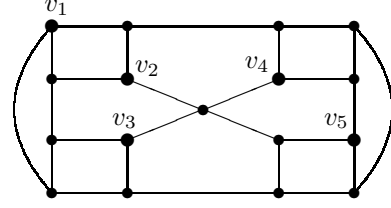
Ten years later, Z. Ouyang, J. Wang and Y. Huang<sup>[9]</sup> studied this parameter too, and obtained that: Let  $G$  be a  $k$ -edge-connected (or  $k$ -connected) simple graph with minimum degree  $\delta$  and girth  $g$ . Then  $\gamma_M(G) \geq \min\{f_k(\delta, g)(\beta(G) + 1), \lfloor \frac{\beta(G)}{2} \rfloor\}$  for  $k = 1, 2, 3$ , where

$\delta$	$f_1(\delta, g)$	$f_2(\delta, g)$	$f_3(\delta, g)$
$\delta = 3$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}(1 - \frac{1}{4^{\lceil \frac{(\delta-2)(\delta+g-3)-3}{4} \rceil + 3}})$
$\delta \geq 4$	$\frac{1}{2}(1 - \frac{3}{4^{\lceil \frac{(\delta-2)(\delta+g-3)-3}{4} \rceil + 1}})$	$\frac{1}{2}(1 - \frac{1}{2^{\lceil \frac{(\delta-2)(\delta+g-3)-3}{4} \rceil + 1}})$	$\frac{1}{2}(1 - \frac{1}{4^{\lceil \frac{(\delta-2)(\delta+g-3)-3}{4} \rceil + 3}})$

There are many examples showing that the lower bound in Theorem B may be best possible. Furthermore, it may be better than the result obtained by Li and Liu<sup>[8]</sup> and the result of Z. Ouyang *etc.*<sup>[9]</sup>. The following are two examples with girth 3 and connectivity 2, and girth 4 and connectivity 3 respectively.



girth 3 and 2-connected



girth 4 and 3-connected

**Fig.10**

In the graph  $G$  depicted in the left of Fig.10, let  $A = \{v_1, v_2\}$ . Then

$$\begin{aligned} & \frac{1}{2} \sum_{i=1}^m (d(v_i) - \varepsilon_i) + \gamma_M(G - \{v_1, v_2, \dots, v_m\}) \\ &= \frac{1}{2}((3-1) + (3-1)) + \gamma_M(G - \{v_1, v_2\}) \\ &= 2 + 2 = 4 = \gamma_M(G) \end{aligned}$$

Obviously, it is bigger than  $\frac{\beta(G)+2}{3}$  ( $= \frac{10}{3}$ ), and is bigger than  $\min\{f_2(\delta, g)(\beta(G) + 1), \lfloor \frac{\beta(G)}{2} \rfloor\}$  ( $= f_2(\delta, g)(\beta(G) + 1) = 3$ ).

In the graph  $G$  depicted in the right of Fig.10, let  $A = \{v_1, v_2, v_3, v_4, v_5\}$ . Then

$$\begin{aligned} & \frac{1}{2} \sum_{i=1}^m (d(v_i) - \varepsilon_i) + \gamma_M(G - \{v_1, v_2, \dots, v_m\}) \\ &= \frac{1}{2}((3-1) \times 5) + \gamma_M(G - \{v_1, v_2, v_3, v_4, v_5\}) \\ &= 5 + 0 = 5 = \gamma_M(G) \end{aligned}$$

Obviously, it is bigger than  $\frac{3\beta(G)+4}{7}$  ( $= \frac{34}{7}$ ), and is bigger than  $\min\{f_3(\delta, g)(\beta(G) + 1), \lfloor \frac{\beta(G)}{2} \rfloor\}$  ( $= f_3(\delta, g)(\beta(G) + 1) = \frac{33}{7}$ ).

#### 4. Independence number and the maximum genus of graphs

Caro[10] and Wei[11] independently shown that for a graph  $G$  its independence number

$$\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{d_G(v) + 1}.$$

Later, Alon and Spencer [12] gave an elegant probabilistic proof of this bound. But, up to now, there is little result concerning the relation between the independence number and the maximum genus of graphs. Let  $N_G(v)$  denote all the neighbors of the vertex  $v$  in  $G$ , the following theorem remedies this deficiency.

**Theorem D** Let  $G = (V, E)$  be a connected 3-regular graph (loops and multi-edges are permitted) with  $A = \{x_1, x_2, \dots, x_{\gamma_M(G)}\}$  be a maximum non-separating independent set of  $G$ . Then its independence number

$$\alpha(G) \geq \gamma_M(G) + \alpha(G - N_A),$$

where  $\alpha(G - N_A)$  is the independence number of the subgraph  $G - N_A$  and  $N_A$  is the *closed closure* of the set  $N_G\{x_1, x_2, \dots, x_{\gamma_M(G)}\}$ , i.e.,  $N_A = (\bigcup_{i=1}^{\gamma_M(G)} N_G(x_i)) \cup \{x_1, x_2, \dots, x_{\gamma_M(G)}\}$ .

**Proof** From Lemma 3.1 we can get that there exists a maximum non-separating independent set  $A = \{x_1, x_2, \dots, x_{\gamma_M(G)}\}$  which satisfies  $G - A$  is connected. Let  $\mathcal{I}$  be an arbitrary independent set of  $G - N_A$ . It is obvious that every vertex in  $A$  is not adjacent to any vertex in  $\mathcal{I}$ . So,  $A \cup \mathcal{I}$  is an independent set of  $G$ , and the theorem is obtained.  $\square$

**Remark** In the graph  $G$  depicted in Fig.11, we may select  $A = \{x_1\}$ . Then  $N_A = \{x_1, x_2, x_6\}$ , and  $\alpha(G - N_A) = 2$ . Noticing  $\alpha(G) = 3$  and  $\gamma_M(G) = 1$ , we can get that  $\alpha(G) = \gamma_M(G) + \alpha(G - N_A) = 3 > \sum_{v \in V(G)} \frac{1}{d_G(v)+1} = \frac{6}{3+1} = \frac{3}{2}$ . So, the lower bound in Theorem D may be best possible, and may be better than that of Caro[10] and Wei[11] in the case of cubic graphs.

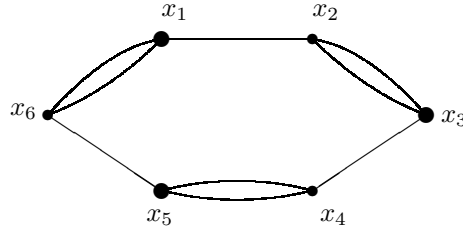


Fig.11

## 5. Estimating the number of the maximum genus embedding of $K_m$

The enumeration of the distinct maximum genus embedding plays an important role in the study of the genus distribution problem, which may be used to decide whether two given graphs are isomorphic. But up to now, except [13] and [14], there is little result concerning the number of the maximum genus embedding of graphs. In this section, we will provide an algorithm to enumerate the number of the distinct maximum genus embedding of the complete graph  $K_m$ , and offer a lower bound which is better than that of S. Stahl<sup>[13]</sup> for  $m \leq 10$ . Furthermore, the enumerative method below can be used to any maximum genus embedding, other than the method in [13] which is restricted to upper embeddable graphs.

A 2-path is called a  $\mathcal{V}$ -type-edge, and is denoted by  $\mathcal{V}$ . If the  $\mathcal{V}$ -type-edge consists of the 2-path  $v_i v_j v_k$ , then this  $\mathcal{V}$ -type-edge is denoted by  $\mathcal{V}_j^{i,k}$  for simplicity. Let  $\psi(G)$  be an embedding of a graph  $G$ . We say that a  $\mathcal{V}$ -type-edge are *inserted* into  $\psi(G)$  if the three

endpoints of the  $\mathcal{V}$ -type-edge are inserted into the corners of the faces in  $\psi(G)$ , yielding an embedding of  $G + \mathcal{V}$ . The following observation can be easily obtained and is essential in this section.

**Observation** Let  $\psi(G)$  be an embedding of a graph  $G$ . We can insert a  $\mathcal{V}$ -type-edge  $\mathcal{V}$  to  $\psi(G)$  to get an embedding  $\rho(G + \mathcal{V})$  of  $G + \mathcal{V}$  so that the face number of  $\rho(G + \mathcal{V})$  is not more than that of  $\psi(G)$ .

**Lemma 5.1** Let  $\psi(G)$  be a *one-face* embedding of the graph  $G$ ,  $v_j$ ,  $v_i$  and  $v_k$  be vertices of  $G$ . If the number of the *face-corner* which containing  $v_j$ ,  $v_i$  and  $v_k$  are  $r_1$ ,  $r_2$  and  $r_3$  respectively, then there are  $r_1 \times r_2 \times r_3$  different ways to add the  $\mathcal{V}$ -type-edge  $\mathcal{V}_j^{i,k}$  to  $\psi(G)$  to get a *one-face* embedding of the graph  $G + \mathcal{V}_j^{i,k}$ .

**Proof** Let the graph depicted in the middle of Fig.12 denote a *one-face* embedding  $\psi(G)$  of the graph  $G$ . Because the number of the *face-corner* which containing  $v_j$ ,  $v_i$  and  $v_k$  are  $r_1$ ,  $r_2$  and  $r_3$  respectively, we can insert the  $\mathcal{V}$ -type-edge  $\mathcal{V}_j^{i,k}$  into  $\psi(G)$  so that there are  $r_1$  different ways to put the edges  $v_j v_k$  and  $v_j v_i$  in the same *face-corner* which containing the vertex  $v_j$ ,  $r_2$  different ways to put the edge  $v_j v_i$  in a *face-corner* which containing the vertex  $v_i$ , and  $r_3$  different ways to put the edge  $v_j v_k$  in a *face-corner* which containing the vertex  $v_k$ . For any one of the  $r_1 \times r_2 \times r_3$  different ways to insert the  $\mathcal{V}$ -type-edge  $\mathcal{V}_j^{i,k}$  into  $\psi(G)$ , we can always get a *one-face* embedding of  $G + \mathcal{V}_j^{i,k}$  by one and only one of the two ways which is depicted by the left and right of Fig.12. So the lemma is obtained.  $\square$

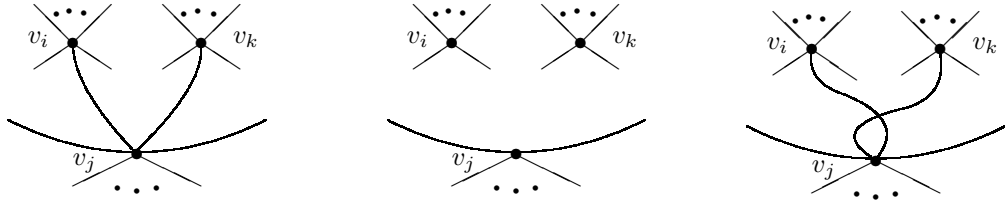


Fig.12

The following algorithm together with Lemma 5.1 provide a maximum genus embedding of  $K_m$  and a lower bound of the number of the maximum genus embedding of  $K_m$ .

### Algorithm

**Note:** Let  $V = \{v_1, v_2, \dots, v_m\}$  be the vertex set of the complete graph  $K_m$ . In the following algorithm,  $\forall i \in \{k, a, b\} \subseteq \{1, 2, \dots, m\}$ , if  $i \equiv 0 \pmod{m}$ , then let  $i = m$ .

**Step 1.** Embed the tree  $v_2 v_3 \dots v_m v_1$  on the plane.

**Step 2.** Let  $k = 1$ ,  $a = 2$ ,  $b = 3$ .

**Step 3.** If the *one-face* embedding of the complete graph  $K_m$  is obtained, then stop. Otherwise, go to Step 4.

**Step 4.** If there are only two vertices  $v_k$  and  $v_a$  that are not adjacent, then connect them to get a *two-face* embedding of the complete graph  $K_m$  and stop. Otherwise, go to Step 5.

**Step 5.** If there is no edge connecting the vertex  $v_k$  and  $v_a$  then go to Step 6. Otherwise, go to Step 10.

**Step 6.** If any pair of  $\{v_k, v_a, v_b\}$  are not the same, and there is no edge connecting the vertex  $v_k$  and  $v_b$  then add the  $\mathcal{V}$ -type-edge  $\mathcal{V}_k^{a,b}$  to the graph to get a *one-face* embedding and go to Step 9. Otherwise, let  $b \equiv b + 1 \pmod{m}$  and go to Step 7.

**Step 7.** If  $b \equiv k - 1 \pmod{m}$  then go to Step 8. Otherwise, go back to Step 6.

**Step 8.** Let  $c = k, k = a, a = c$  (i.e., exchange  $k$  and  $a$ ). Then go back to Step 3.

**Step 9.** Let  $b \equiv a + 3 \pmod{m}$ ,  $a \equiv a + 2 \pmod{m}$ , and go to Step 11.

**Step 10.** Let  $a \equiv a + 1 \pmod{m}$ , and go to Step 11.

**Step 11.** If  $a \equiv k - 1 \pmod{m}$ , then go to Step 12. Otherwise, go back to Step 3.

**Step 12.** Let  $k = 1$ , and go to Step 13.

**Step 13.** If  $d_G(v_k) < m - 1$ , then let  $a \equiv k + 2 \pmod{m}$ ,  $b \equiv k + 3 \pmod{m}$ , and go back to Step 3. Otherwise, go to Step 14.

**Step 14.** If  $d_G(v_k) = m - 1$ , then let  $k = k + 1$ , and go back to Step 13.

Using the above algorithm, we can get the maximum genus embedding of  $K_m$  except that  $m = 1 + 8i$  or  $m = 6 + 8i$  ( $i = 0, 1, 3, \dots$ ). Furthermore, for  $m \leq 10$ , our result is much better than that of Stahl<sup>[13]</sup>. For simplicity, we give some symbols which are used below. Let  $E$  be a *one-face* embedding of a graph. Then the symbol  $(\mathcal{V}_j^{i,k} : r_1 \times r_2 \times r_3)$  means that there are  $r_1 \times r_2 \times r_3$  different ways to add the  $\mathcal{V}$ -type-edge  $\mathcal{V}_j^{i,k}$  to  $E$  to get a *one-face* embedding of  $E + \mathcal{V}_j^{i,k}$ , and the symbol  $(e_j^{j,k} : r_1 \times r_2)$  means that there are  $r_1 \times r_2$  different ways to add the edge  $v_j v_k$  to  $E$  to get a *two-face* embedding of  $E + v_j v_k$ .

**Result 1** The number of the maximum genus embedding of the complete graph  $K_8$  is at least  $2^{26} \times 3^{11} \times 5^5$ .

**Proof** Let  $V = \{v_1, v_2, \dots, v_8\}$  be the vertex set of the complete graph  $K_8$ . There is only one way to embed the tree  $T = v_2 v_3 \dots v_8 v_1$  on the plane, which is a *one-face* embedding, and is denoted by  $\mathcal{E}_1$ . In  $\mathcal{E}_1$ , the number of the *face-corner* which containing the vertex  $v_1, v_2, v_3$  is 1, 1 and 2 respectively. So, according to Lemma 5.1, there are 2 different ways to add the  $\mathcal{V}$ -type-edge  $\mathcal{V}_1^{2,3}$  to  $\mathcal{E}_1$  to get a *one-face* embedding of  $T + \mathcal{V}_1^{2,3}$ . Let  $\mathcal{E}_2$  be any one of the *one-face* embedding of  $T + \mathcal{V}_1^{2,3}$ . In  $\mathcal{E}_2$ , the number of the *face-corner* which containing the vertex  $v_1, v_4, v_5$  is 3, 2 and 2 respectively. So, according to Lemma 5.1, there are  $3 \times 2 \times 2 (= 12)$  different ways to add the  $\mathcal{V}$ -type-edge  $\mathcal{V}_1^{4,5}$  to  $\mathcal{E}_2$  to get a *one-face* embedding of  $T + \mathcal{V}_1^{2,3} + \mathcal{V}_1^{4,5}$ . Similarly, we can get that for each of the *one-face* embedding of  $T + \mathcal{V}_1^{2,3} + \mathcal{V}_1^{4,5}$ , there are  $5 \times 2 \times 2$  different ways to add the  $\mathcal{V}$ -type-edge  $\mathcal{V}_1^{6,7}$  to  $T + \mathcal{V}_1^{2,3} + \mathcal{V}_1^{4,5}$  to get a *one-face* embedding of  $T + \mathcal{V}_1^{2,3} + \mathcal{V}_1^{4,5} + \mathcal{V}_1^{6,7}$ .

Similarly, we can add  $\mathcal{V}$ -type-edges, one by one in the following order, to  $T + \mathcal{V}_1^{2,3} + \mathcal{V}_1^{4,5} + \mathcal{V}_1^{6,7}$  to get a *two-face* embedding of  $K_8$  eventually.

$(\mathcal{V}_2^{4,5} : 2 \times 3 \times 3), (\mathcal{V}_2^{6,7} : 4 \times 3 \times 3), (\mathcal{V}_8^{2,3} : 2 \times 6 \times 3), (\mathcal{V}_8^{4,5} : 4 \times 4 \times 4), (\mathcal{V}_6^{8,3} : 4 \times 6 \times 4),$   
 $(\mathcal{V}_4^{6,7} : 5 \times 6 \times 4), (\mathcal{V}_3^{5,7} : 5 \times 5 \times 5), (e_5^{5,7} : 6 \times 6).$

So, the number of the distinct maximum genus embedding of  $K_8$  is at least

$$\begin{aligned} & 2 \times (3 \times 2 \times 2) \times (5 \times 2 \times 2) \times (2 \times 3 \times 3) \times (4 \times 3 \times 3) \times (2 \times 6 \times 3) \\ & \times (4 \times 4 \times 4) \times (4 \times 6 \times 4) \times (5 \times 6 \times 4) \times (5 \times 5 \times 5) \times (6 \times 6) \\ & = 2^{26} \times 3^{11} \times 5^5 \end{aligned}$$

**Result 2** The number of the distinct maximum genus embedding of the complete graph  $K_{10}$  is at least  $2^{52} \times 3^{15} \times 5^7 \times 7^6$ , which is obtained from the unique *one-face* embedding of the tree  $T = v_2 v_3 \dots v_{10} v_1$  by successively adding the following  $\mathcal{V}$ -type-edges:  $(\mathcal{V}_1^{2,3} : 1 \times 1 \times 2), (\mathcal{V}_1^{4,5} : 3 \times 2 \times 2), (\mathcal{V}_1^{6,7} : 5 \times 2 \times 2), (\mathcal{V}_1^{8,9} : 7 \times 2 \times 2),$   
 $(\mathcal{V}_2^{4,5} : 2 \times 3 \times 3), (\mathcal{V}_2^{6,7} : 4 \times 3 \times 3), (\mathcal{V}_2^{8,9} : 6 \times 3 \times 3), (\mathcal{V}_{10}^{2,3} : 2 \times 8 \times 3), (\mathcal{V}_{10}^{4,5} : 4 \times 4 \times 4),$   
 $(\mathcal{V}_{10}^{6,7} : 6 \times 4 \times 4), (\mathcal{V}_8^{10,3} : 4 \times 8 \times 4), (\mathcal{V}_8^{4,5} : 6 \times 5 \times 5), (\mathcal{V}_6^{8,9} : 5 \times 8 \times 4), (\mathcal{V}_6^{3,4} : 7 \times 5 \times 6),$   
 $(\mathcal{V}_3^{5,7} : 6 \times 6 \times 5), (\mathcal{V}_9^{3,4} : 5 \times 8 \times 7), (\mathcal{V}_9^{5,7} : 7 \times 7 \times 6), (\mathcal{V}_7^{4,5} : 7 \times 8 \times 8).$

**Result 3** The number of the distinct maximum genus embedding of the complete graph  $K_7$  is at least 49766400000, which is obtained from the unique *one-face* embedding of the tree  $T = v_2 v_3 \dots v_7 v_1$  by successively adding the following  $\mathcal{V}$ -type-edges:  $(\mathcal{V}_1^{2,3} : 1 \times 1 \times 2), (\mathcal{V}_1^{4,5} : 3 \times 2 \times 2), (\mathcal{V}_6^{1,2} : 2 \times 5 \times 2), (\mathcal{V}_6^{3,4} : 4 \times 3 \times 3), (\mathcal{V}_2^{4,5} : 3 \times 4 \times 3),$   
 $(\mathcal{V}_7^{2,3} : 2 \times 5 \times 4), (\mathcal{V}_7^{4,5} : 4 \times 5 \times 4), (e_3^{3,5} : 5 \times 5).$

**Result 4** The number of the distinct maximum genus embedding of the complete graph  $K_5$  is at least 432, which is obtained from the unique *one-face* embedding of the tree  $T = v_2 v_3 v_4 v_5 v_1$  by successively adding the following  $\mathcal{V}$ -type-edges:  $(\mathcal{V}_1^{2,3} : 1 \times 1 \times 2),$   
 $(\mathcal{V}_4^{1,2} : 2 \times 3 \times 2), (\mathcal{V}_5^{2,3} : 2 \times 3 \times 3).$

The algorithm doesn't work for  $K_6$  and  $K_9$ . But the maximum genus embedding of  $K_6$  and  $K_9$  can be obtained by the following manners.

**Result 5** The number of the distinct maximum genus embedding of the complete graph  $K_6$  is at least 663552, which is obtained from the unique *one-face* embedding of the tree  $T = v_2 v_3 \dots v_6 v_1$  by successively adding the following  $\mathcal{V}$ -type-edges:  $(\mathcal{V}_1^{2,3} : 1 \times 1 \times 2),$   
 $(\mathcal{V}_1^{4,5} : 3 \times 2 \times 2), (\mathcal{V}_2^{4,5} : 2 \times 3 \times 3), (\mathcal{V}_6^{2,4} : 2 \times 4 \times 4), (\mathcal{V}_3^{5,6} : 3 \times 4 \times 4).$

**Result 6** The number of the distinct maximum genus embedding of the complete graph  $K_9$  is at least  $2^{27} \times 3^{12} \times 5^7 \times 7^6$ , which is obtained from the unique *one-face* embedding of the tree  $T = v_2 v_3 \dots v_9 v_1$  by successively adding the following  $\mathcal{V}$ -type-edges:  $(\mathcal{V}_1^{2,3} : 1 \times 1 \times 2), (\mathcal{V}_1^{4,5} : 3 \times 2 \times 2), (\mathcal{V}_1^{6,7} : 5 \times 2 \times 2), (\mathcal{V}_8^{1,2} : 2 \times 7 \times 2), (\mathcal{V}_8^{3,4} : 4 \times 3 \times 3),$   
 $(\mathcal{V}_8^{5,6} : 6 \times 3 \times 3), (\mathcal{V}_2^{4,5} : 3 \times 4 \times 4), (\mathcal{V}_2^{6,7} : 5 \times 4 \times 3), (\mathcal{V}_9^{2,3} : 2 \times 7 \times 4), (\mathcal{V}_9^{4,5} : 4 \times 5 \times 5),$   
 $(\mathcal{V}_9^{6,7} : 6 \times 5 \times 4), (\mathcal{V}_3^{5,6} : 5 \times 6 \times 6), (\mathcal{V}_7^{3,5} : 5 \times 7 \times 7), (\mathcal{V}_4^{6,7} : 6 \times 7 \times 7).$

**Remark** Saul Stahl<sup>[13]</sup> obtained that the complete graph  $K_m$  on  $m$  vertices has at least  $[(m-6)!]^4 [(m-3)!]^{m-4}$  maximum genus embeddings, and for  $m \equiv 0, 3 \pmod{4}$   $K_m$

has at least  $(\frac{m-2}{m-1})^2[(m-3)!]^m$  maximum genus embeddings. It is obvious that our results for  $m \leq 10$  is much better than that of Stahl.

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